Pythagoras' theorem and trigonometric functions

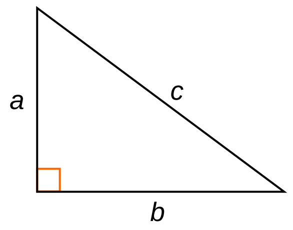
Book 2 Ch 1 p.9
Skill: mathematics

a Pythagoras' theorem

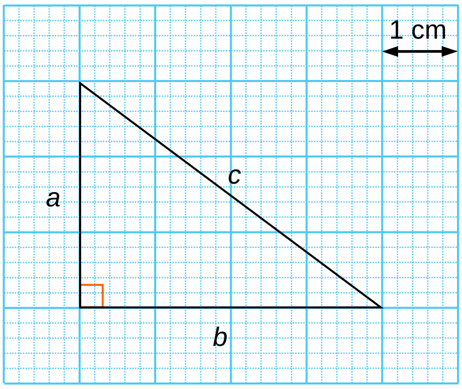
Pythagoras' theorem states that for any right-angled triangle, the three sides are related by the following equation:

$$c^2 = a^2 + b^2$$

Therefore, by using this equation, if the lengths of two sides of the triangle are known, the length of the third side could be found.



Consider a right-angled triangle with a = 3 cm and b = 4 cm.



By Pythagoras' theorem,

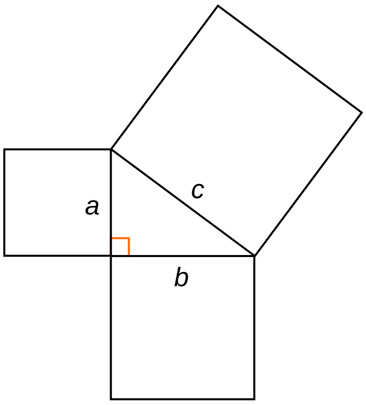
$$c^2 = a^2 + b^2$$

$$c = \sqrt{a^2 + b^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \text{ cm}$$

You can verify this result by measuring c with a ruler.

More to know…

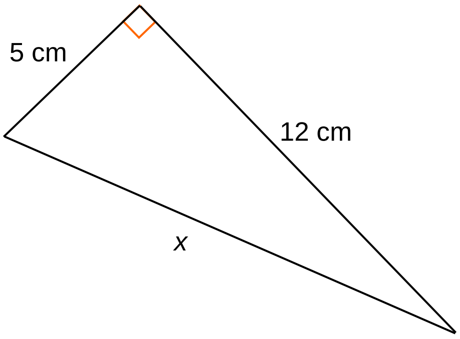
Pythagoras' theorem was named after the ancient Greek mathematician Pythagoras who was perhaps the first to give a proof of the theorem. Pythagoras showed that, for the squares on each side of a right-angled triangle, the area of the largest square is the sum of the area of the other two squares.



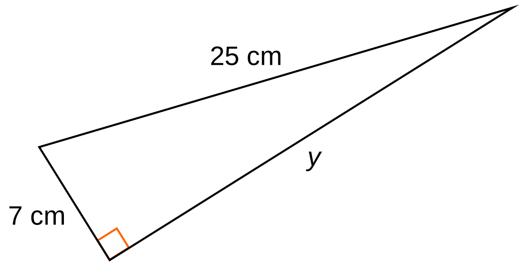
Example

Find the unknown in each of the following right-angled triangles.

(a)



(b)



Solution

(a) By Pythagoras' theorem,

$$x^2 = 5^2 + 12^2$$

$$x = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13 \text{ cm}$$

(b) By Pythagoras' theorem,

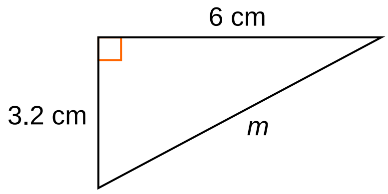
$$25^2 = 7^2 + y^2$$

$$y = \sqrt{25^2 - 7^2} = \sqrt{625 - 49} = \sqrt{576} = 24 \text{ cm}$$

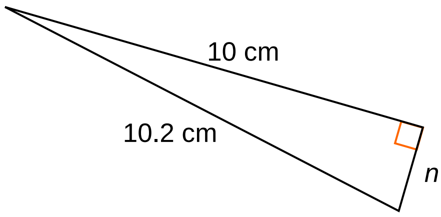
Exercise 1

1 Find the unknown in each of the following right-angled triangles.

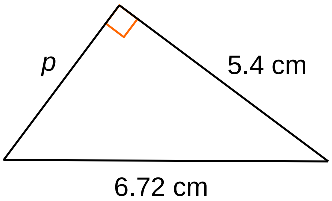
(a)



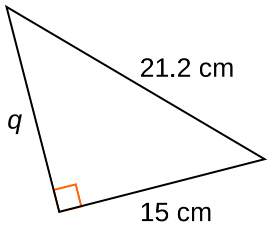
(b)



(c)



(d)



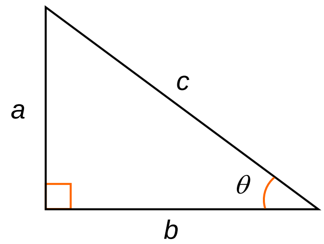
b Trigonometric functions

Consider the right-angled triangle shown. The trigonometric functions are defined as follows.

The sine (正弦) of angle θ: $\sin\theta = \frac{a}{c}$

The cosine (餘弦) of angle θ: $\cos\theta = \frac{b}{c}$

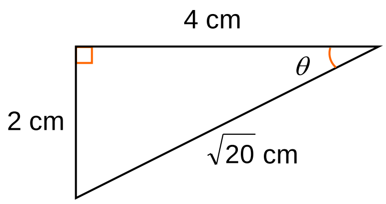
The tangent (正切) of angle θ: $\tan\theta = \frac{a}{b}$



Note that trigonometric functions do not have any unit.

Example

Given a right-angled triangle, find **(a)** $\sin\theta$; **(b)** $\cos\theta$; and **(c)** $\tan\theta$.



Solution

From the triangle, a = 2 cm, b = 4 cm and $c = \sqrt{20}$ cm.

(a) $\sin\theta = \frac{a}{c} = \frac{2}{\sqrt{20}} = 0.447$

(b) $\cos\theta = \frac{b}{c} = \frac{4}{\sqrt{20}} = 0.894$

(c) $\tan\theta = \frac{a}{b} = \frac{2}{4} = \frac{1}{2} = 0.5$

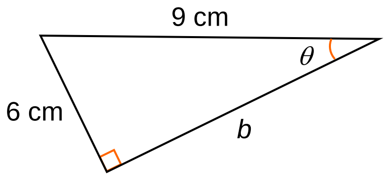
Example

Consider the right-angled triangle as shown. Find

(a) $\sin\theta$;

(b) $\cos\theta$; and

(c) $\tan\theta$.



Solution

From the figure, a = 6 cm and c = 9 cm. Side b can be found by Pythagoras' theorem.

$$c^2 = a^2 + b^2 \Rightarrow b = \sqrt{c^2 - a^2} = \sqrt{9^2 - 6^2} = \sqrt{45} = 3\sqrt{5} \text{ cm}$$

(a) $\sin\theta = \frac{a}{c} = \frac{6}{9} = \frac{2}{3} = 0.667$

(b) $\cos\theta = \frac{b}{c} = \frac{3\sqrt{5}}{9} = 0.745$

(c) $\tan\theta = \frac{a}{b} = \frac{6}{3\sqrt{5}} = 0.894$

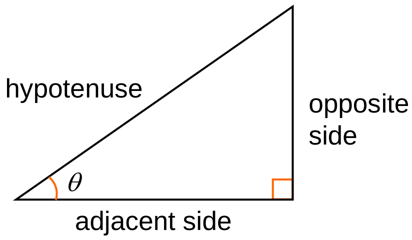
More to know…

In a right-angled triangle with an acute angle θ, the sides of triangle are named as follows:

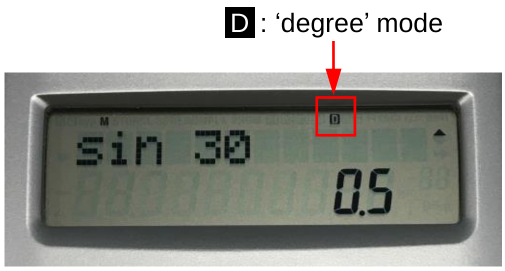
Hypotenuse (斜邊): the longest side in the triangle which is opposite the right angle

Adjacent side (鄰邊) of θ: the side next to θ (but not the hypotenuse)

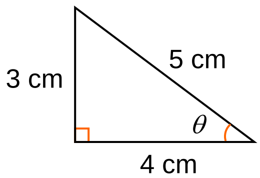
Opposite side (對邊) of θ: the side opposite to θ

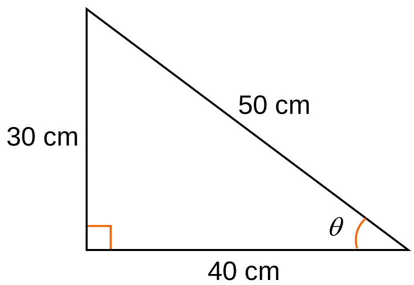


If the angle θ in a right-angled triangle is known, the trigonometric functions can be easily found by using a scientific calculator. For example, if $\theta = 30°$, sin 30° = 0.5. (Remember to set the calculator in 'degree' mode first.)



The sine ratio is always equal to 0.5 for any right-angled triangle with $\theta = 30°$. From this, we can see that the trigonometric functions for any two triangles with the same θ will remain the same no matter how big or small the triangles are. The ratios depend on θ only.





$$\sin\theta = \frac{3}{5} \qquad \sin\theta = \frac{30}{50} = \frac{3}{5}$$

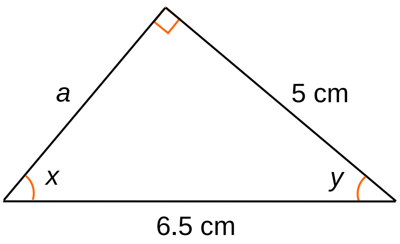
$$\cos\theta = \frac{4}{5} \qquad \cos\theta = \frac{40}{50} = \frac{4}{5}$$

$$\tan\theta = \frac{3}{4} \qquad \tan\theta = \frac{30}{40} = \frac{3}{4}$$

Conversely, we can find the angle θ if one of the trigonometric functions is given. This can also be done by using a scientific calculator.

Example

Find the unknowns in the following figure.



Solution

By Pythagoras' theorem,

$$a^2 = 6.5^2 - 5^2$$

$$a = \sqrt{6.5^2 - 5^2} = 4.15 \text{ cm}$$

$$\sin x = \frac{5}{6.5} \quad \Rightarrow \quad x = 50.3°$$

$$\cos y = \frac{5}{6.5} \quad \Rightarrow \quad y = 39.7°$$

More to know…

In the above example, we need to find the inverse of a sine ratio. We can use the following expression.

$$x = \sin^{-1}\left(\frac{5}{6.5}\right) = 50.3°$$

The symbol $\sin^{-1}$ means the inverse sine. We should note that $\sin^{-1} x$ is NOT equal to $\frac{1}{\sin x}$.

Press SHIFT + [sin] when finding x using a scientific calculator.

The symbol for the inverse of other trigonometric functions are as follows:

The inverse cosine: $\cos^{-1}$

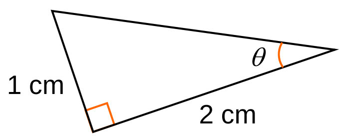
The inverse tangent: $\tan^{-1}$

More examples on finding the unknown angles or unknown length of the sides of a right-angled triangle are shown on the next page.

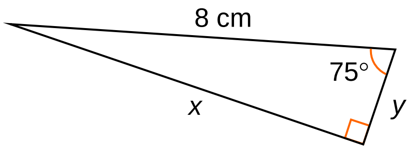
Example

Find the unknown(s) in each of the following figures.

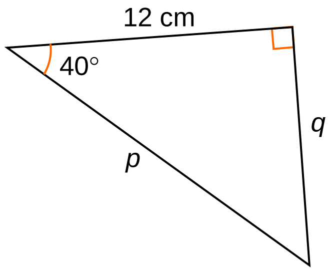
(a)



(b)



(c)



Solution

(a) $\tan\theta = \dfrac{1}{2} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\dfrac{1}{2}\right) = 26.6°$

(b) $\sin 75° = \dfrac{x}{8} \quad \Rightarrow \quad x = 8 \sin 75° = 7.73 \text{ cm}$

$\cos 75° = \dfrac{y}{8} \quad \Rightarrow \quad y = 8 \cos 75° = 2.07 \text{ cm}$

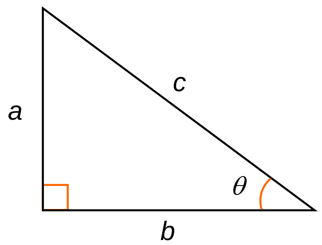
(c) $\cos 40° = \dfrac{12}{p} \quad \Rightarrow \quad p = \dfrac{12}{\cos 40°} = 15.7 \text{ cm}$

$\tan 40° = \dfrac{q}{12} \quad \Rightarrow \quad q = 12 \tan 40° = 10.1 \text{ cm}$

Note that for a right-angled triangle, $a < c$ and $b < c$. Therefore,

$$\sin\theta = \frac{a}{c} < 1$$

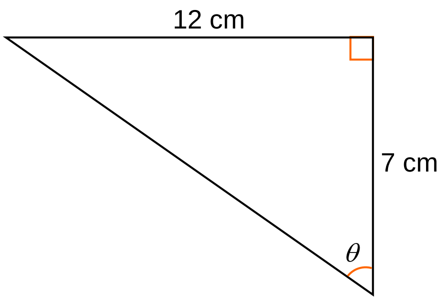
$$\cos\theta = \frac{b}{c} < 1$$



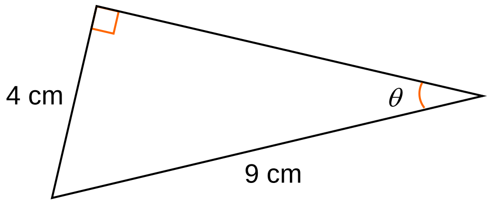
Exercise 2

1 Find **(i)** sin θ, **(ii)** cos θ and **(iii)** tan θ for each of the following right-angled triangles.

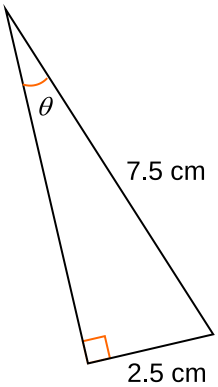
(a)



(b)



(c)



2 Find the following using a scientific calculator.

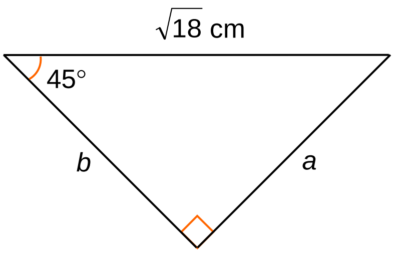
(a) **(i)** sin 45° **(ii)** cos 45° **(iii)** tan 45°

(b) **(i)** sin 75° **(ii)** cos 75° **(iii)** tan 75°

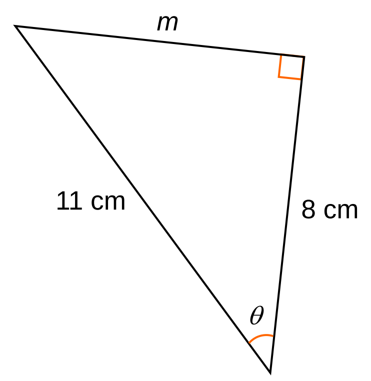
(c) **(i)** $\sin^{-1} 0.25$ **(ii)** $\cos^{-1} 0.5$ **(iii)** $\tan^{-1} 1.25$

3 Find the unknowns in each of the following right-angled triangles.

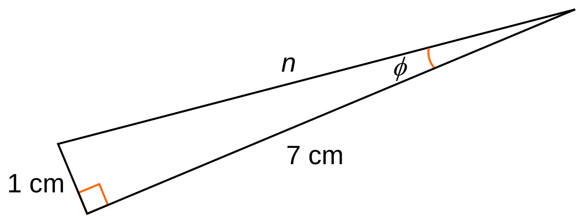
(a)



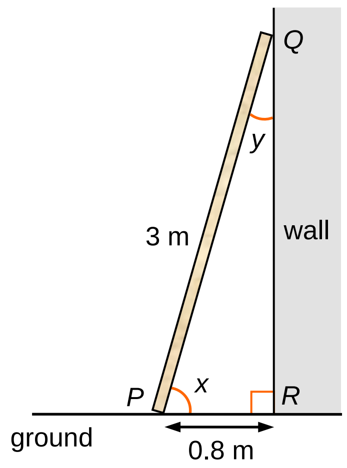
(b)



(c)



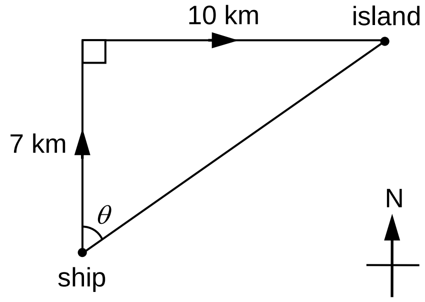
4 A ladder of 3 m long is put on a horizontal ground against a vertical wall. The base P of the ladder is 0.8 m from the wall.



(a) What is the height of the top Q of the ladder?

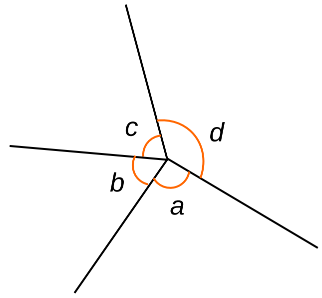
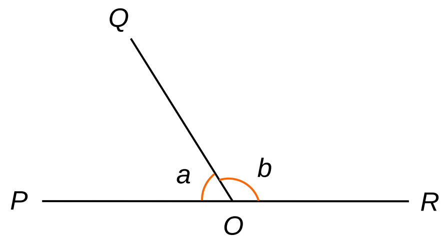
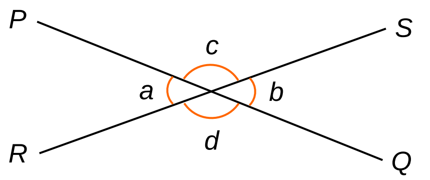
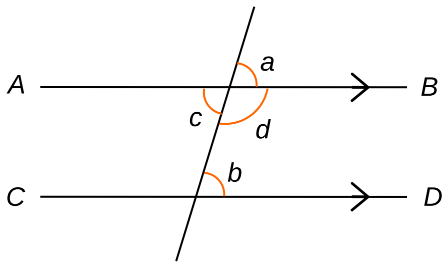
(b) Find angles x and y.

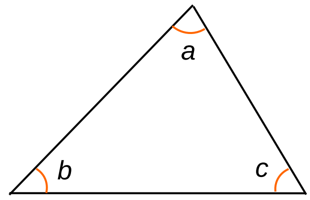
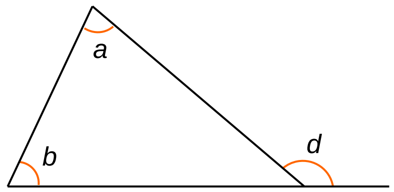
5 A ship sails 7 km due north and then sails 10 km due east to an island. Find the total displacement of the ship.



C Finding an angle using geometry

The following shows some properties in geometry that are needed in finding angles.

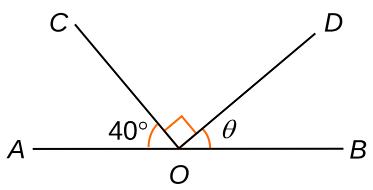
Property	Example
∠s at a pt. The sum of angles at a point is 360°.	 $a + b + c + d = 360°$ (∠s at a pt.)
adj. ∠s on st. line The sum of adjacent angles on a straight line is 180°.	 $a + b = 180°$ (adj. ∠s on st. line)
vert. opp. ∠s When two straight lines intersect, the vertically opposite angles formed are equal.	 $a = b$ (vert. opp. ∠s) $c = d$ (vert. opp. ∠s)
Consider the case when two parallel lines (*AB* and *CD*) are cut by another straight line.	
corr. ∠s, *AB* // *CD* The corresponding angles are equal.	$a = b$ (corr. ∠s, *AB* // *CD*)
alt. ∠s, *AB* // *CD* The alternate angles are equal.	$b = c$ (alt. ∠s, *AB* // *CD*)
int. ∠s, *AB* // *CD* The sum of interior angles is 180°.	$b + d = 180°$ (int. ∠s, *AB* // *CD*)

∠ sum of △ The angles in a triangle add up to 180°.	 $a + b + c = 180°$ (∠ sum of △)
ext. ∠ of △ An exterior angle of a triangle is equal to the sum of its two interior opposite angles.	 $d = a + b$ (ext. ∠ of △)

Example

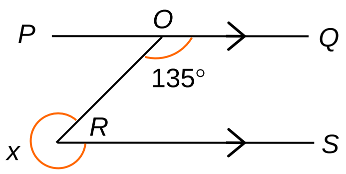
Find the unknown angle in each of the following figures.

(a)



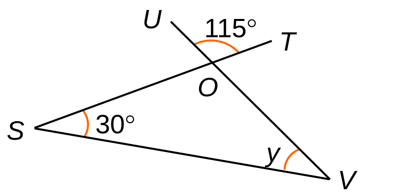
AOB is a straight line.

(b)



POQ is a straight line.

(c)



SOT and *UOV* are straight lines.

Solution

(a) $40° + 90° + \theta = 180°$ (adj. ∠s on st. line)

$\theta = 50°$

(b) $135° + \angle ORS = 180°$ (int. ∠s, PQ // RS)

$\angle ORS = 45°$

$\angle ORS + x = 360°$ (∠s at a pt.)

$45° + x = 360°$

$x = 315°$

(c) $\angle SOV = 115°$ (vert. opp. ∠s)

$30° + \angle SOV + y = 180°$ (∠ sum of △)

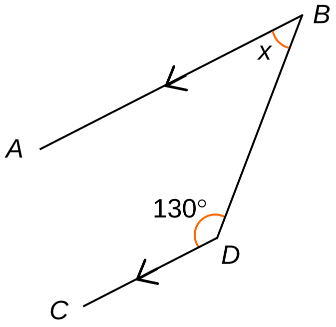
$30° + 115° + y = 180°$

$y = 35°$

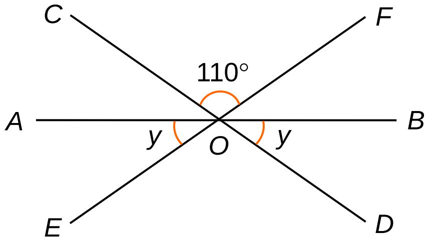
Exercise 3

1 Find the unknown angles in the following figures.

(a)

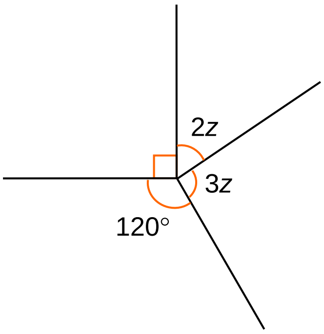


(b)

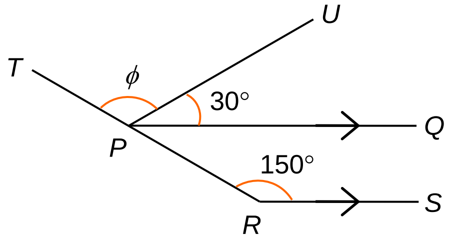


AOB, *COD* and *EOF* are straight lines.

(c)

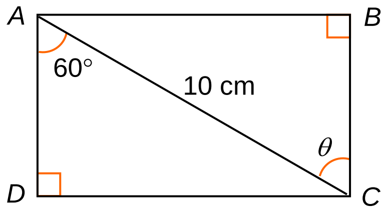


(d)



TPR is a straight line.

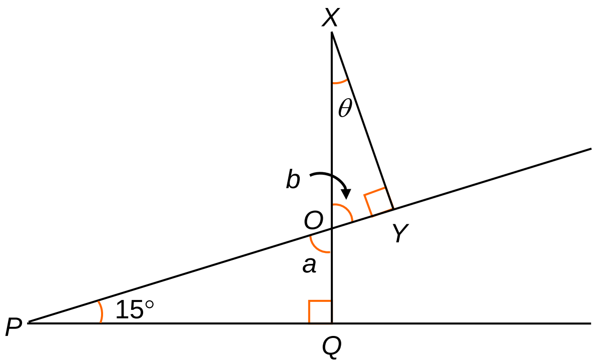
2 Consider the figure as shown. *ABCD* is a rectangle.



(a) Find *CD*.

(b) Find θ, and hence find *BC*.

3 Consider the figure as shown.



(a) By filling in the blanks, show that $\theta = 15°$.

In △*OPQ*,

15° + *a* + 90° = _________ (∠ sum of △)

a = _________

Besides, *b* = *a* = _________ (___________________________)

In △*OXY*,

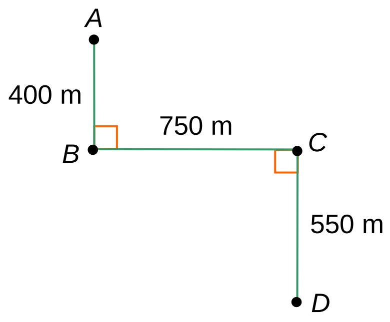
θ + *b* + 90° = 180° (___________________________)

θ + _________ + 90° = 180°

θ = _________

(b) Given *OX* = 10 cm, find *OY* and *XY*.

4 In an orienteering game, Jenny starts at checkpoint *A*, passes checkpoints *B* and *C*, and arrives at checkpoint *D*.



(a) What is the shortest distance between checkpoints *A* and *D*?

(b) Find the direction of *D* from *A*. Express your answer in reduced bearing.

Answers

Exercise 1

1 **(a)** 6.8 cm
(b) 2.01 cm
(c) 4.00 cm
(d) 15.0 cm

Exercise 2

1 **(a)** **(i)** 0.864
(ii) 0.504
(iii) 1.71
(b) **(i)** 0.444
(ii) 0.896
(iii) 0.496
(c) **(i)** 0.333
(ii) 0.943
(iii) 0.354

2 **(a)** **(i)** 0.707 **(ii)** 0.707 **(iii)** 1
(b) **(i)** 0.966 **(ii)** 0.259 **(iii)** 3.73
(c) **(i)** 14.5° **(ii)** 60° **(iii)** 51.3°

3 **(a)** a = 3 cm; b = 3 cm
(b) m = 7.55 cm; θ = 43.3°
(c) ϕ = 8.13°; n = 7.07 cm

4 **(a)** 2.89 m
(b) x = 74.5°; y = 15.5°

5 12.2 km (N55.0°E)

Exercise 3

1 **(a)** 50°
(b) 35°
(c) 30°
(d) 120°

2 **(a)** 8.66 cm
(b) 60°; 5 cm

3 **(a)** In $\triangle OPQ$,

$15° + a + 90° = 180°$ (∠ sum of △)

$a = 75°$

Besides, $b = a = 75°$ (vert. opp. ∠s)

In $\triangle OXY$,

$\theta + b + 90° = 180°$ (∠ sum of △)

$\theta + 75° + 90° = 180°$

$\theta = 15°$

(b) OY = 2.59 cm; XY = 9.66 cm

4 **(a)** 1210 m
(b) S38.3°E